We consider skew scattering in this note taking single gate screened Coulomb potential with charge 1 as example.

$$V_i(|r-R_i|)=\frac{e^2}{4\pi\varepsilon_0\varepsilon}\left(\frac{1}{|r-R_i|}-\frac{1}{\sqrt{|r-R_i|^2+d^2}}\right)$$

Here $\frac{d}{2}$ is the distance between TBG and gate. In plane wave basis, we have the scattering Hamiltonian $H_i$

$$H_i=\frac{1}{\Omega}\sum_{k,k'}\int dr\, e^{i(k-k')\cdot r}V_i(|r-R_i|)\; c_{k'}^{\dagger}c_k$$

$$=\frac{1}{\Omega}\sum_{k,q}e^{iq\cdot R_i}V_i(q)c_k^{\dagger}c_{k+q}$$

Here $e^{iq\cdot R_i}V_i(q)=e^{iq\cdot R_i}\int dr\, e^{iq\cdot(r-R_i)}V_i(|r-R_i|)\;=e^{iq\cdot R_i}\frac{e^2}{2\varepsilon_0\varepsilon}\frac{1}{q}(1-e^{-qd})$ and $\Omega$ is the total area. We would like to rewrite this term in band basis to finish computation of scattering matrix

$$H_i=\frac{1}{\Omega}\sum_{k,q,K,Q}e^{i(q+Q)\cdot R_i}V_i(q+Q)c_{k+K}^{\dagger}c_{k+q+K+Q}$$

$$=\frac{1}{\Omega}\sum_{k,q,K,Q}e^{i(q+Q)\cdot R_i}V_i(q+Q)\sum_m u_m^*(k+K)c_{k,m}^{\dagger}\sum_n u_n(k+q+K+Q)c_{k+q,n}$$

$$=\frac{1}{\Omega}\sum_{k,q,Q}e^{i(q+Q)\cdot R_i}V_i(q+Q)\sum_{m,n,K}u_m^*(k+K)u_n(k+q+K+Q)\,c_{k,m}^{\dagger}c_{k+q,n}$$

$$=\frac{1}{\Omega}\sum_{k,q,Q}e^{i(q+Q)\cdot R_i}V_i(q+Q)\sum_{m,n}\lambda_{m,n}(k,k+q+Q)\,c_{k,m}^{\dagger}c_{k+q,n}$$

Here we use unitary transformation $c_{k+K}^{\dagger}=\sum_m u_m^*(k+K)c_{k,m}^{\dagger}$ and define the form factor $\lambda_{m,n}(k,k+q+Q)=\sum_K u_m^*(k+K)u_n(k+q+K+Q)$. Only considering flat band part and sum over all scattering center $R_i$, one can extract the scattering potential matrix $V_{k,m;k+q,n}$

$$V_{k,m;k+q,n}=\frac{1}{\Omega}\sum_{R_i,Q}e^{i(q+Q)\cdot R_i}V_i(q+Q)\lambda_{m,n}(k,k+q+Q)$$

According to Lippmann-Schwinger formula and Born approximation, we keep the scattering matrix $T$ up to the second order of $V$

$$T=V+V\frac{1}{\epsilon_{k,m}-H_0+i\mu}V$$

For symmetric contribution of scattering rate, we have Fermi's golden rule with leading order $O(V^2)$

$$w_{k,m,k+q,n}^{(S)}=\frac{2\pi}{\hbar}\left|V_{k,m;k+q,n}\right|^2\delta(\epsilon_{k,m}-\epsilon_{k+q,n})$$

But this symmetric scattering will not contribute to skew scattering, one need to compute antisymmetric scattering rate $w_{k,m;k+q,n}^{(A)}$ with leading order $O(V^3)$

$$w_{k,m;k+q,n}^{(A)}=\frac{i2\pi^2}{\hbar}\sum_{p,l}\langle V_{k,m;k+q,n}V_{k+p,l;k,m}V_{k+q,n;k+p,l}-c.c.\rangle_{dis}\delta(\epsilon_{k,m}-\epsilon_{k+p,l})\,\delta(\epsilon_{k,m}-\epsilon_{k+q,n})$$

$$=-\frac{(2\pi)^2}{\hbar}\sum_{p,l}Im(\langle V_{k,m;k+q,n}V_{k+p,l;k,m}V_{k+q,n;k+p,l}\rangle_{dis})\delta(\epsilon_{k,m}-\epsilon_{k+p,l})\,\delta(\epsilon_{k,m}-\epsilon_{k+q,n})$$

$$= -\frac{(2\pi)^2}{\hbar} \sum_{p,l\in(\epsilon_{k,m}\pm\triangle)} \frac{Im\left(\langle V_{k,m;k+q,n} V_{k+p,l;k,m} V_{k+q,n;k+p,l}\rangle_{dis}\right)}{2\,\triangle} \delta\left(\epsilon_{k,m} - \epsilon_{k+q,n}\right)$$

Here $\langle\ \ \rangle_{dis}$ is the distribution average for scattering center, $\triangle$ is small and the summation only for states which within the energy region $\epsilon_{k,m} \pm \triangle$ and we have used the equation below to derive the formula above

$$\lim_{\mu\to 0} Im\left(\frac{1}{\epsilon - H_0 + i\mu}\right) = \lim_{\mu\to 0} P \int_{-\infty}^{\infty} \frac{\delta(\epsilon-\epsilon')}{\epsilon' - H_0 + i\mu} d\epsilon' = i\pi\delta(\epsilon - H_0)$$

We assume different scattering centers are random distribution and keep the leading order of impurity density $\frac{N_i}{\Omega}$ then the contribution is direct production keeping momentum conservation. Ignore side jump, and finally we can put this $w^{(A)}_{k,m;k+q,n}$ in the skew scattering conductance coming from Boltzmann equation

$$\partial_t f_l + qE_\alpha e^{iwt}\partial_\alpha f_l = -\sum_{l'} (w_{l'l} f_l - w_{ll'} f_{l'})$$

Here $l$ is the label for state with momentum $k$ and band $m$. We expand $w_{ll'}$ and $f_l$ up to leading order by seeing antisymmetric scattering rate $w^{(A)}_{k,m;k+q,n}$ and electric field $\boldsymbol{E} = E_\alpha e^{iwt}$ as perturbation

$$w_{ll'} = w^{(S)}_{ll'} + w^{(A)}_{ll'}$$

$$f_l = f_l^{(0,0)} + f_l^{(1,0)} + f_l^{(1,1)} + f_l^{(2,0)} + f_l^{(2,1)}$$

Here $f_l^{(n_1,n_2)}$ means distribution function of order $E^{n_1}$ and $\left(w^{(A)}\right)^{n_2}$ and $f_l^{(0,0)} = \frac{1}{e^{\beta(\epsilon-\mu)}+1}$ is fermi distribution. By noticing $\sum_{l'} w^{(A)}_{ll'} = 0$, We have recursive relation

$$\partial_t f_l^{(1,0)} + qE_\alpha e^{iwt}\partial_\alpha f_l^{(0,0)} = -\sum_{l'} w^{(S)}_{l'l}\left(f_l^{(1,0)} - f_{l'}^{(1,0)}\right)$$

$$\partial_t f_l^{(1,1)} = -\sum_{l'} w^{(S)}_{l'l}\left(f_l^{(1,1)} - f_{l'}^{(1,1)}\right) + \sum_{l'} w^{(A)}_{ll'} f_{l'}^{(1,0)}$$

$$\partial_t f_l^{(2,0)} + qE_\alpha e^{iwt}\partial_\alpha f_l^{(1,0)} = -\sum_{l'} w^{(S)}_{l'l}\left(f_l^{(2,0)} - f_{l'}^{(2,0)}\right)$$

$$\partial_t f_l^{(2,1)} + qE_\alpha e^{iwt}\partial_\alpha f_l^{(1,1)} = -\sum_{l'} w^{(S)}_{l'l}\left(f_l^{(2,1)} - f_{l'}^{(2,1)}\right) + \sum_{l'} w^{(A)}_{ll'} f_{l'}^{(2,0)}$$

Take relaxation time approximation

$$\sum_{l'} w^{(S)}_{l'l}\left(f_l^{(n_1,n_2)} - f_{l'}^{(n_1,n_2)}\right) = \frac{f_l^{(n_1,n_2)}}{\tau^{(n_1,n_2)}}$$

We can solve recursive equations self-consistently

$$f_l^{(1,0)} = \frac{-q\tau^{(1,0)}}{1 + iw\tau^{(1,0)}} E_\alpha e^{iwt}\partial_\alpha f_l^{(0,0)}$$

$$f_l^{(1,1)} = \frac{\tau^{(1,1)}}{1 + iw\tau^{(1,1)}} \sum_{l'} w^{(A)}_{ll'} f_{l'}^{(1,0)}$$

$$f_l^{(2,0)} = \frac{-q\tau^{(2,0)}}{1 + i2w\tau^{(2,0)}} E_\alpha e^{iwt}\partial_\alpha f_l^{(1,0)}$$

$$f_l^{(2,1)} = \frac{\tau^{(2,1)}}{1 + i2w\tau^{(2,1)}} \sum_{l'} w^{(A)}_{ll'} f_{l'}^{(2,0)} + \frac{-q\tau^{(2,1)}}{1 + i2w\tau^{(2,1)}} E_\alpha e^{iwt}\partial_\alpha f_l^{(1,1)}$$

According to the definition of current operator and velocity operator

$$j_\alpha = q\int v_\alpha f = \sigma_{\alpha\beta}E_\beta + \chi_{\alpha\beta\gamma}E_\beta E_\gamma$$

$$v_\alpha = \partial_\alpha \epsilon - q(\boldsymbol{E}\times\boldsymbol{\Omega})_\alpha$$

The dominant contribution from skew scattering is

$$\sigma_{\alpha\beta}^{(skew)} = \frac{q\tau^{(1,1)}}{1+iw\tau^{(1,1)}}\frac{-q\tau^{(1,0)}}{1+iw\tau^{(1,0)}}\sum_{l_m}\int d^2k\,\frac{\partial\epsilon_l}{\partial k_\alpha}\sum_{l'} w_{ll'}^{(A)}\partial'_\beta f_{l'}^{(0,0)}$$

$$\chi_{\alpha\beta\gamma}^{(skew,1)} = \frac{q\tau^{(2,1)}}{1+i2w\tau^{(2,1)}}\frac{-q\tau^{(2,0)}}{1+iw\tau^{(2,0)}}\frac{-q\tau^{(1,0)}}{1+iw\tau^{(1,0)}}\sum_{l_m}\int d^2k\,\frac{\partial\epsilon_l}{\partial k_\alpha}\sum_{l'} w_{ll'}^{(A)}\partial'_\beta\partial'_\gamma f_{l'}^{(0,0)}$$

$$\chi_{\alpha\beta\gamma}^{(skew,2)} = \frac{-q^2\tau^{(2,1)}}{1+i2w\tau^{(2,1)}}\frac{\tau^{(1,1)}}{1+iw\tau^{(1,1)}}\frac{-q\tau^{(1,0)}}{1+iw\tau^{(1,0)}}\sum_{l_m}\int d^2k\,\frac{\partial\epsilon_l}{\partial k_\alpha}\partial_\beta\sum_{l'} w_{ll'}^{(A)}\partial'_\gamma f_{l'}^{(0,0)}$$

Here $l_m$ means the summation to band label in $l$. If we assume $w\to 0$ and $\tau^{(n_1,n_2)}\to\tau$, we have

$$\sigma_{\alpha\beta}^{(skew)} = \frac{(\pi q\tau)^2}{\hbar}\sum_m\int d^2k\,\frac{\partial\epsilon_m(k)}{\partial k_\alpha}\sum_{p,l;q,n\in(\epsilon_{k,m}\pm\triangle)}\frac{Im\big(V_{k,m;k+q,n}V_{k+p,l;k,m}V_{k+q,n;k+p,l}\big)}{\triangle^2}\partial'_\beta f_n^{(0,0)}(k+q)$$

$$\chi_{\alpha\beta\gamma}^{(skew,1)} = -\frac{\pi^2(q\tau)^3}{\hbar}\sum_m\int d^2k\,\frac{\partial\epsilon_m(k)}{\partial k_\alpha}\sum_{p,l;q,n\in(\epsilon_{k,m}\pm\triangle)}\frac{Im\big(V_{k,m;k+q,n}V_{k+p,l;k,m}V_{k+q,n;k+p,l}\big)}{\triangle^2}\partial'_\beta\partial'_\gamma f_n^{(0,0)}(k+q)$$

$$\chi_{\alpha\beta\gamma}^{(skew,2)} = -\frac{\pi^2(q\tau)^3}{\hbar}\sum_m\int d^2k\,\frac{\partial\epsilon_m(k)}{\partial k_\alpha}\partial_\beta\sum_{p,l;q,n\in(\epsilon_{k,m}\pm\triangle)}\frac{Im\big(V_{k,m;k+q,n}V_{k+p,l;k,m}V_{k+q,n;k+p,l}\big)}{\triangle^2}\partial'_\gamma f_n^{(0,0)}(k+q)$$

Here $\partial'_\beta$ means the partial derivative at band $n$ and momentum $k+q$. These expressions are the form easy to compute numerically. One can notice only bands near fermi surface will contribute according to $\partial'_\beta f_n^{(0,0)}(k+q)$. We take $m$ as flat bands in TBG. By setting an energy window $\triangle$, for a given $k$ we compute $\frac{\partial\epsilon_m(k)}{\partial k_\alpha}$ and collect all states within the energy window. By iterating collected states, we can compute the rest part in the integral.