Q1: What is conformal transformation?

A1: Coordinate transformation preserving angle.

$$C = \frac{v \cdot w}{\sqrt{v^2 w^2}} = \frac{v_\alpha w^\alpha}{\sqrt{v^\beta v_\beta w^\gamma w_\gamma}}$$

$$\rightarrow \frac{g'_{\mu\upsilon}(x')v'^{\mu}(x')w'^{\upsilon}(x')}{\sqrt{g'_{\alpha\beta}(x')g'_{\gamma\delta}(x')\left(v'^{\alpha}(x')v'^{\beta}(x')w'^{\gamma}(x')w'^{\delta}(x')\right)}} = \frac{g_{\mu\upsilon}(x)v^{\mu}(x)w^{\upsilon}(x)}{\sqrt{g_{\alpha\beta}(x)g_{\gamma\delta}(x)\left(v^{\alpha}(x)v^{\beta}(x)w^{\gamma}(x)w^{\delta}(x)\right)}}$$

$$\frac{g'_{\mu\upsilon}(x')}{\sqrt{g'_{\alpha\beta}(x')g'_{\gamma\delta}(x')}} = \frac{g_{\mu\upsilon}(x)}{\sqrt{g_{\alpha\beta}(x)g_{\gamma\delta}(x)}}$$

$$\boxed{g'_{\mu\upsilon}(x') = \Omega(x)g_{\mu\upsilon}(x)}$$

Q2: For minimal conformal transformation $x^\mu \rightarrow x'^\mu = x^\mu + \epsilon^\mu$ in flat space, which requirements must be there for ϵ^μ?

A2: For flat space,

$$g_{\alpha\beta}(x) = \begin{bmatrix} \pm 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \pm 1 \end{bmatrix}$$

Since we require line element must be unchanged by any coordinate transformation, after a minimal coordinate transformation $x^\mu \rightarrow x'^\mu = x^\mu + \epsilon^\mu(x)$ we have

$$ds^2 = g_{\alpha\beta}(x)dx^\alpha dx^\beta = g'_{\alpha\beta}(x')dx'^\alpha dx'^\beta$$

$$= g'_{\alpha\beta}(x')\frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\upsilon}dx^\mu dx^\upsilon = g'_{\alpha\beta}(x')\left(\delta^\alpha_\mu + \partial_\mu \epsilon^\alpha\right)\left(\delta^\beta_\upsilon + \partial_\upsilon \epsilon^\beta\right)dx^\mu dx^\upsilon$$
$$= \Omega(x)\left(g_{\mu\upsilon}(x) + \partial_\mu \epsilon_\upsilon + \partial_\upsilon \epsilon_\mu + \partial_\mu \epsilon_\beta \partial_\upsilon \epsilon^\beta\right)dx^\mu dx^\upsilon$$

so that

$$\frac{\left(1 - \Omega(x)\right)}{\Omega(x)}g_{\mu\upsilon}(x) = \partial_\mu \epsilon_\upsilon + \partial_\upsilon \epsilon_\mu + \partial_\mu \epsilon_\beta \partial_\upsilon \epsilon^\beta \approx \partial_\mu \epsilon_\upsilon + \partial_\upsilon \epsilon_\mu$$

For infinitesimal ϵ^{μ}, now we have

$$\boxed{\partial_\mu \epsilon_\upsilon + \partial_\upsilon \epsilon_\mu = f(x) g_{\mu\upsilon}}$$

Use $g^{\mu\upsilon}$ contracts all indexes,

$$\boxed{f(x) = \frac{2}{D}\partial_\mu \epsilon^{\mu} = \frac{2}{D}\partial \cdot \epsilon}$$

Here D is dimension of $g_{\mu\upsilon}$. Besides, we apply ∂_ρ to both side of $\partial_\mu \epsilon_\upsilon + \partial_\upsilon \epsilon_\mu = f(x) g_{\mu\upsilon}$,

$$\begin{cases} \partial_\rho \partial_\mu \epsilon_\upsilon + \partial_\rho \partial_\upsilon \epsilon_\mu = \partial_\rho f g_{\mu\upsilon} \\ \partial_\mu \partial_\upsilon \epsilon_\rho + \partial_\mu \partial_\rho \epsilon_\upsilon = \partial_\mu f g_{\upsilon\rho} \\ \partial_\upsilon \partial_\rho \epsilon_\mu + \partial_\upsilon \partial_\mu \epsilon_\rho = \partial_\upsilon f g_{\rho\mu} \end{cases}$$

(2)+(3)-(1),

$$2\partial_\mu \partial_\upsilon \epsilon_\rho = \partial_\upsilon f g_{\rho\mu} + \partial_\mu f g_{\upsilon\rho} - \partial_\rho f g_{\mu\upsilon}$$

Use $g^{\mu\upsilon}$ contracts all indexes,

$$2\partial^2 \epsilon_\rho = (2-D)\partial_\rho f$$

Apply ∂_μ for $2\partial^2 \epsilon_\rho = (2-D)\partial_\rho f$, and ∂^2 for $\partial_\mu \epsilon_\upsilon + \partial_\upsilon \epsilon_\mu = f(x) g_{\mu\upsilon}$

$$2\partial^2 \partial_\mu \epsilon_\rho = (2-D)\partial_\mu \partial_\rho f$$

$$\partial^2 \partial_\mu \epsilon_\upsilon + \partial^2 \partial_\upsilon \epsilon_\mu = \boxed{(2-D)\partial_\mu \partial_\upsilon f = \partial^2 f g_{\mu\upsilon}}$$

Again, use $g^{\mu\upsilon}$ contracts all indexes,

$$\boxed{(D-1)\partial^2 f = 0}$$

It is very interesting that for $D > 1$, there must be $\partial^2 f = 0$, so that $(2-D)\partial_\mu \partial_\upsilon f = 0$. Again, if $D > 2$, one more limitation $\partial_\mu \partial_\upsilon f = 0$ means $f(x) = A + B_\alpha x^\alpha$. Finally, we require

$$\boxed{\epsilon^\mu = a^\mu + b^\mu{}_\alpha x^\alpha + c^\mu{}_{\beta\gamma} x^\beta x^\gamma}$$

Here according to symmetry, $c^\mu{}_{\beta\gamma} = c^\mu{}_{\gamma\beta}$. Let's discuss those coefficients in detail.

1. For a^μ, one can see it is just infinitesimal translation.
2. For $b^\mu{}_\alpha x^\alpha$, according to $\partial_\mu \epsilon_\upsilon + \partial_\upsilon \epsilon_\mu = f(x) g_{\mu\upsilon}$

$$b_{\upsilon\mu} + b_{\mu\upsilon} = \frac{2}{D} b^\alpha{}_\alpha g_{\mu\upsilon}$$

   This means $b_{\mu\alpha}$ has two parts, one for diagonal elements and the other for off-diagonal anti-symmetrical elements $b_{\mu\alpha} = \lambda g_{\mu\alpha} + m_{\mu\alpha}$. Here $m_{\mu\alpha} = -m_{\alpha\mu}$. One can see λx^μ is for infinitesimal dilatation and $m^\mu{}_\alpha x^\alpha$ is for infinitesimal rotation.

3. Finally, for $c^\mu{}_{\beta\gamma} x^\beta x^\gamma$, according to $\partial_\mu \epsilon_\upsilon + \partial_\upsilon \epsilon_\mu = f(x) g_{\mu\upsilon}$

$$c_{\mu\upsilon\gamma} x^\gamma + c_{\mu\beta\upsilon} x^\beta + c_{\upsilon\mu\gamma} x^\gamma + c_{\upsilon\beta\mu} x^\beta = \frac{2g_{\mu\upsilon}}{D}\left(c^\alpha{}_{\alpha\gamma} x^\gamma + c^\alpha{}_{\beta\alpha} x^\beta\right)$$

$$c_{\mu\upsilon\gamma}x^{\gamma} + c_{\upsilon\mu\gamma}x^{\gamma} = \frac{2g_{\mu\upsilon}}{D}c^{\alpha}{}_{\alpha\gamma}x^{\gamma}$$

$$\begin{cases} c_{\mu\upsilon\gamma} + c_{\upsilon\mu\gamma} = \frac{2g_{\mu\upsilon}}{D}c^{\alpha}{}_{\alpha\gamma} \\ c_{\upsilon\gamma\mu} + c_{\gamma\upsilon\mu} = \frac{2g_{\upsilon\gamma}}{D}c^{\alpha}{}_{\alpha\mu} \\ c_{\gamma\mu\upsilon} + c_{\mu\gamma\upsilon} = \frac{2g_{\gamma\mu}}{D}c^{\alpha}{}_{\alpha\upsilon} \end{cases}$$

(2)+(3)-(1),

$$c_{\gamma\mu\upsilon} = \frac{g_{\gamma\mu}}{D}c^{\alpha}{}_{\alpha\upsilon} + \frac{g_{\upsilon\gamma}}{D}c^{\alpha}{}_{\alpha\mu} - \frac{g_{\mu\upsilon}}{D}c^{\alpha}{}_{\alpha\gamma} = g_{\gamma\mu}d_{\upsilon} + g_{\upsilon\gamma}d_{\mu} - g_{\mu\upsilon}d_{\gamma}$$

Here we just define $d_{\mu} = \frac{c^{\alpha}{}_{\alpha\mu}}{D}$. One can see after this infinitesimal 'special conformal transformation' (SCT), $x'^{\mu} = x^{\mu} + c^{\mu}{}_{\beta\gamma}x^{\beta}x^{\gamma} = x^{\mu} + 2d_{\gamma}x^{\gamma}x^{\mu} - d^{\mu}x^{\beta}x_{\beta} = x^{\mu} + 2(d\cdot x)x^{\mu} - d^{\mu}x^{2}$

Q3: Can we find generate operators according to infinitesimal conformal transformation?

A3: Yes. The construction is straightforward. Since after infinitesimal transformation, we have $x'^{\mu} = e^{i\hat{\tau}}x^{\mu} = x^{\mu} + i\hat{\tau}x^{\mu}$

1. For translation, $\hat{\tau} = -ia^{\alpha}\partial_{\alpha}$.
2. For rotation (boost), $\hat{\tau} = -im^{\alpha}{}_{\beta}x^{\beta}\partial_{\alpha}$. Here $m^{\alpha}{}_{\beta} = -m_{\beta}{}^{\alpha}$.
3. For dilatation, $\hat{\tau} = -i\lambda x^{\alpha}\partial_{\alpha}$
4. For special conformal transformation, $\hat{\tau} = -id^{\alpha}\left(2x_{\alpha}x^{\beta}\partial_{\beta} - x^{2}\partial_{\alpha}\right)$

Q4: Can we find finite conformal transformation according to generators we derive?

A4: Yes. Just expand $e^{i\hat{\tau}} = \sum_{n=0}^{\infty}\frac{1}{n!}(i\hat{\tau})^{n}$ and apply it to x^{μ}.

1. For translation, $x'^{\mu} = e^{i\hat{\tau}}x^{\mu} = x^{\mu} + a^{\mu}$
2. For rotation (boost), $x'^{\mu} = e^{i\hat{\tau}}x^{\mu} = e^{m^{\mu}{}_{\beta}}x^{\beta}$
3. For dilatation, $x'^{\mu} = e^{i\hat{\tau}}x^{\mu} = e^{\lambda}x^{\mu}$

4. For special conformal transformation, it is not so straightforward. We can define $x''^{\mu} = \frac{x^{\mu}}{x^2}$ first, so that $x''^2 = \frac{1}{x^2}$ and $x^{\mu} = \frac{x''^{\mu}}{x''^2}$. Then one can see $\hat{\tau} = -id^{\alpha}\left(2x_{\alpha}x^{\beta}\partial_{\beta} - x^2\partial_{\alpha}\right) = -id^{\alpha}\left(2x_{\alpha}x^{\beta}\frac{\partial x''^{\gamma}}{\partial x^{\beta}}\partial''_{\gamma} - x^2\frac{\partial x''^{\gamma}}{\partial x^{\alpha}}\partial''_{\gamma}\right)$.

It can be calculated that $\frac{\partial x''^{\gamma}}{\partial x^{\beta}} = \frac{\delta^{\gamma}_{\beta}x^2 - 2x^{\gamma}x_{\beta}}{x^4}$. Plug this in $\hat{\tau}$,

$$\hat{\tau} = -id^{\alpha}\left(2x_{\alpha}x^{\beta}\frac{\delta^{\gamma}_{\beta}x^2 - 2x^{\gamma}x_{\beta}}{x^4}\partial''_{\gamma} - x^2\frac{\delta^{\gamma}_{\alpha}x^2 - 2x^{\gamma}x_{\alpha}}{x^4}\partial''_{\gamma}\right)$$

$$= -id^{\alpha}\left(2x_{\alpha}x^{\beta}\left(\delta^{\gamma}_{\beta}x''^2 - 2x^{\gamma}x_{\beta}x''^4\right)\partial''_{\gamma} - \left(\delta^{\gamma}_{\alpha} - 2x^{\gamma}x_{\alpha}x''^2\right)\partial''_{\gamma}\right)$$

$$= -id^{\alpha}(-\partial''_{\alpha})$$

One can see $e^{i\hat{\tau}}x''^{\mu} = x''^{\mu} - d^{\mu}$, which means $\frac{x'^{\mu}}{x'^2} = \frac{x^{\mu}}{x^2} - d^{\mu}$. By noticing $\frac{1}{x'^2} = \frac{1}{x^2} + d^2 - 2\frac{d\cdot x}{x^2}$, we finally have $x'^{\mu} = e^{i\hat{\tau}}x^{\mu} = \frac{x^{\mu} - d^{\mu}x^2}{1 + d^2x^2 - 2d\cdot x}$.

Q5: Is there any straightforward understanding about special conformal transformation?

A5: Yes. Below, we can see SCT is just two discrete inversions inserted a translation between.

First, we will prove discrete inversion $x'^{\mu} = \frac{x^{\mu}}{x^2}$ is also a conformal transformation, which means we need prove

$$\boxed{\Omega(x)g_{\alpha\beta}(x)\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial x'^{\beta}}{\partial x^{\upsilon}} = g_{\mu\upsilon}(x)}$$

As one can see

$$g_{\alpha\beta}\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial x'^{\beta}}{\partial x^{\upsilon}} = g_{\alpha\beta}\frac{\delta^{\alpha}_{\mu}x^2 - 2x^{\alpha}x_{\mu}}{x^4}\frac{\delta^{\beta}_{\upsilon}x^2 - 2x^{\beta}x_{\upsilon}}{x^4} = \frac{g_{\mu\upsilon}}{x^4}$$

So that here $\Omega(x) = x^4$, $x'^{\mu} = \frac{x^{\mu}}{x^2}$ is also a conformal transformation.

Now, we follow ‘inversion-translation-inversion’ steps

Inversion: $x'^{\mu} = \frac{x^{\mu}}{x^2}$

Translation: $x''^{\mu} = x'^{\mu} - d^{\mu}$

Inversion: $x'''^{\mu} = \frac{x''^{\mu}}{x''^2} = \frac{x'^{\mu} - d^{\mu}}{x'^2 + d^2 - 2d\cdot x'} = \frac{\frac{x^{\mu}}{x^2} - d^{\mu}}{\frac{1}{x^2} + d^2 - 2\frac{d\cdot x}{x^2}} = \frac{x^{\mu} - d^{\mu}x^2}{1 + d^2x^2 - 2d\cdot x}$

Q6: What is 'primary field'? What does the correlation function look like for primary field?

A6: The most straightforward definition of 'primary field' is the field under conformal transformation changes like

$$\phi(x) \to \phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta}{D}} \phi(x')$$

Here Δ is so called scaling dimension of the field $\phi(x)$, Jacobian determinant $\left|\frac{\partial x'}{\partial x}\right|$ can be expressed by $\Omega(x)$ according to $\Omega(x) g_{\alpha\beta}(x) \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} = g_{\mu\nu}(x)$:

$$\left|\frac{\partial x'}{\partial x}\right| = \Omega^{-\frac{D}{2}}$$

Then according to conformal transformation, we can see for primary field

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\frac{\Delta_1}{D}} \left|\frac{\partial x'}{\partial x}\right|_{x=x_2}^{\frac{\Delta_2}{D}} \langle \phi_1(x_1')\phi_2(x_2')\rangle$$

As we all know, Jacobian determinant for translation or rotation (boost) is equal to 1, so that this will give a limitation $\langle \phi_1(x_1)\phi_2(x_2)\rangle = f(|x_1 - x_2|)$. For dilatation, $\langle \phi_1(x_1)\phi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2}\langle \phi_1(\lambda x_1)\phi_2(\lambda x_2)\rangle$, so that this give one more limitation $f(|x_1 - x_2|) = \lambda^{\Delta_1+\Delta_2} f(\lambda|x_1 - x_2|)$. This actually means $\boxed{\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{d_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}}$, here d_{12} has nothing to do with x_1, x_2 and is only determined by ϕ_1, ϕ_2.

Finally, for SCT, we would like to derive Jacobian determinant $\left|\frac{\partial x'}{\partial x}\right|$ first, which follows 'three steps' way in A5.

$$\left|\frac{\partial x'''}{\partial x}\right| = \left|\frac{\partial x'''}{\partial x''}\right| \left|\frac{\partial x''}{\partial x'}\right| \left|\frac{\partial x'}{\partial x}\right| = \frac{1}{x''^{2D}} \frac{1}{x^{2D}} = \left(\frac{1}{x^2\left(\frac{1}{x^2} + d^2 - 2\frac{d\cdot x}{x^2}\right)}\right)^D = \frac{1}{(1 + d^2x^2 - 2d\cdot x)^D} \equiv \frac{1}{\gamma^D}$$

And $|x_1' - x_2'| = \sqrt{\left(\frac{x_1^{\mu} - d^{\mu}x_1^2}{1 + d^2x_1^2 - 2d\cdot x_1} - \frac{x_2^{\mu} - d^{\mu}x_2^2}{1 + d^2x_2^2 - 2d\cdot x_2}\right)^2} = \sqrt{\frac{x_1^2}{1 + d^2x_1^2 - 2d\cdot x_1} + \frac{x_2^2}{1 + d^2x_2^2 - 2d\cdot x_2} - 2\frac{x_1\cdot x_2 + d^2x_1^2x_2^2 - x_2^2 d\cdot x_1 - x_1^2 d\cdot x_2}{(1 + d^2x_1^2 - 2d\cdot x_1)(1 + d^2x_2^2 - 2d\cdot x_2)}} = \sqrt{\frac{(x_1 - x_2)^2}{(1 + d^2x_1^2 - 2d\cdot x_1)(1 + d^2x_2^2 - 2d\cdot x_2)}} = \frac{|x_1 - x_2|}{\gamma_1^{\frac{1}{2}}\gamma_2^{\frac{1}{2}}}$

Then one can see SCT requires

$$\frac{d_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}}=\frac{1}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}}\frac{d_{12}}{|x_1'-x_2'|^{\Delta_1+\Delta_2}}=\frac{\gamma_1^{\frac{\Delta_1+\Delta_2}{2}}\gamma_2^{\frac{\Delta_1+\Delta_2}{2}}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}}\frac{d_{12}}{|x_1-x_2|^{\Delta_1+\Delta_2}}$$

Since this equation is true for any SCT γ, so we require $\frac{\Delta_1+\Delta_2}{2}=\Delta_1$ which means $\Delta_1=\Delta_2$, or $\langle\phi_1(x_1)\phi_2(x_2)\rangle=0$ for $\Delta_1\neq\Delta_2$.

Similarly, we can see three-point correlation $\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle=f(|x_1-x_2|,|x_2-x_3|,|x_1-x_3|)$ by translation and rotation (boost), and $f(|x_1-x_2|,|x_2-x_3|,|x_1-x_3|)=\lambda^{\Delta_1+\Delta_2+\Delta_3}f(\lambda|x_1-x_2|,\lambda|x_2-x_3|,\lambda|x_1-x_3|)$ by dilatation. Without SCT, we require

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle=\frac{C_{123}}{|x_1-x_2|^a|x_2-x_3|^b|x_1-x_3|^c}$$

Here $a+b+c=\Delta_1+\Delta_2+\Delta_3$, C_{123} is only determined by ϕ_1, ϕ_2, ϕ_3.
SCT again gives limitation

$$1=\frac{(\gamma_1\gamma_2)^{\frac{a}{2}}(\gamma_2\gamma_3)^{\frac{b}{2}}(\gamma_1\gamma_3)^{\frac{c}{2}}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}\gamma_3^{\Delta_3}}$$

This requires $a+c=2\Delta_1$, $a+b=2\Delta_2$ and $b+c=2\Delta_3$, which means $a=\Delta_1+\Delta_2-\Delta_3$, $b=\Delta_2+\Delta_3-\Delta_1$ and $c=\Delta_1+\Delta_3-\Delta_2$. Explicitly,

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle=\frac{C_{123}}{|x_1-x_2|^{(\Delta_1+\Delta_2)-\Delta_3}|x_2-x_3|^{(\Delta_2+\Delta_3)-\Delta_1}|x_1-x_3|^{(\Delta_1+\Delta_3)-\Delta_2}}$$

For four-point correlation, one can see $\frac{r_{12}r_{34}}{r_{13}r_{24}}$ and $\frac{r_{12}r_{34}}{r_{14}r_{23}}$ $(r_{ij}=\left|x_i-x_j\right|)$ are also unchanged under all conformal transformation. So, four-point correlation form cannot be determined only by conformal symmetry. Generally, it should have the form $\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle=F\left(\frac{r_{12}r_{34}}{r_{13}r_{24}},\frac{r_{12}r_{34}}{r_{14}r_{23}}\right)\frac{1}{r_{12}^a r_{13}^b r_{14}^c r_{23}^d r_{24}^e r_{34}^f}$, here $a+b+c+d+e+f=\Delta_1+\Delta_2+\Delta_3+\Delta_4$ and $a+b+c=2\Delta_1$, $a+d+e=2\Delta_2$, $b+d+f=2\Delta_3$, $c+e+f=2\Delta_4$. Since we cannot determine 6 variables by 4 independent equations, there is also freedom for coefficients.