Exact solution of 2D Ising model on square and honeycomb lattice

Xu Zhang

Introduction with 2D square lattice

We would like to solve a 2D Ising model in square lattice first. The Hamiltonian of this model is, (Assuming $J_1, J_2 > 0$)

$$H = -\sum_{i=1}^{m}\sum_{j=1}^{n}(J_1\sigma_{i,j}^z\sigma_{i+1,j}^z + J_2\sigma_{i,j}^z\sigma_{i,j+1}^z)$$

Since we would like to see thermodynamical property like temperature of phase transition of this model. We need to find out the expression of partition function, which is,

$$Z = \sum_{\{\sigma^z\}} e^{-\beta H}$$

By rewriting $K_i = \beta J_i$, we have

$$Z = \sum_{\{\sigma^z\}}\prod_{i=1}^{m} e^{\sum_{j=1}^{n}(K_1\sigma_{i,j}^z\sigma_{i+1,j}^z + K_2\sigma_{i,j}^z\sigma_{i,j+1}^z)}$$

The reason we write partition function in this way is we want to separate index m out as a contracted index of matrixes and reduce this dimension by transfer matrix method,

$$Z = \sum_{\{\sigma^z\}}\left\langle 1\left|e^{\sum_{j=1}^{n}(K_1\sigma_{1,j}^z\sigma_{2,j}^z + K_2\sigma_{1,j}^z\sigma_{1,j+1}^z)}\right|2\right\rangle$$

$$\cdot\left\langle 2\left|e^{\sum_{j=1}^{n}(K_1\sigma_{2,j}^z\sigma_{3,j}^z + K_2\sigma_{2,j}^z\sigma_{2,j+1}^z)}\right|3\right\rangle$$

$$\cdots$$

$$\cdot\langle m|\cdots|1\rangle$$

(For convenient, we use periodical boundary condition in this direction. As for other boundary condition, the result should be similar as we get below)

Here $\langle 1|A_{12}|2\rangle$ means elements of matrix A_{12} can be determined by $2^n \times 2^n$ combinations of variables $\sigma_{1,j}^z$ and $\sigma_{2,j}^z$.

If we define $2^n \times 2^n$ matrixes V_1, V_2 as below,

$$V_1 = e^{\sum_{j=1}^{n} K_1\sigma_j^z\sigma'^z_j}$$

$$V_2 = e^{\sum_{j=1}^{n} K_2\sigma_j^z\sigma_{j+1}^z}\prod_{i=1}^{n}\delta_{\sigma_i^z,\sigma'^z_i}$$

We can rewrite partition function in a simple way,

$$Z = Tr[(V_1V_2)^m]$$

Now this problem become much easier to solve. Just notice that, the trace of one matrix does not change after we diagonalize this matrix. What we need to do for next step is diagonalize matrix V_1V_2, and $Z = \sum_{i=1}^{2^n} \lambda_i^m$.(Here λ_i is the eigenvalue of V_1V_2 labeled by i)

Diagonalization

Before we diagonalize the product of those 2 matrixes, we would like to study those matrixes one by one. Since the matrix which can diagonalize e^A is just the matrix that diagonalize A, we will focus on exponential part of each V_i matrix.

Here we use a small trick,

$$e^{K\sigma_j^z \sigma'^z_j} = \begin{bmatrix} e^K & e^{-K} \\ e^{-K} & e^K \end{bmatrix} = \begin{bmatrix} Acosh(K^*) & Asinh(K^*) \\ Asinh(K^*) & Acosh(K^*) \end{bmatrix} = Ae^{K^*\sigma_j^x}$$

To make this equation correct, there are two relations needs to be satisfied,

$$\tanh(K^*) = e^{-2K}$$

$$A = \frac{e^K}{\cosh(K^*)} = \frac{e^{-K}}{\sinh(K^*)} = \sqrt{\frac{2}{\sinh(2K^*)}} = \sqrt{2\sinh(2K)}$$

We see the exponential part of V_i matrixes as Hamiltonians, by using the trick we introduced above, those 2 Hamiltonian are,

$$H_1 = \sum_{j=1}^{n} K_1^* \sigma_j^x$$

$$H_2 = \sum_{j=1}^{n} K_2 \sigma_j^z \sigma_{j+1}^z$$

Remember we should try our best to diagonalize them in the same basis. We will use Jordan Wigner transformation, Fourier transformation and Bogoliubov transformation to do this job.

According to routine of Jordan Wigner transformation, we define,

$$S^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$S^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

This definition will make,

$$S^+ \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$S^- \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$S^+ \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

$$S^{-}\begin{bmatrix}0\\1\end{bmatrix} = 0$$

Besides, in basis diagonalize σ^x,

$$\sigma^x = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} = 1 - 2S^{-}S^{+}$$

$$\sigma^z = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = S^{-} + S^{+}$$

It is easy to prove by using matrix form,

$$S^{-}\sigma^x = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} = -\sigma^x S^{-}$$

$$S^{+}\sigma^x = \begin{bmatrix}0 & -1\\0 & 0\end{bmatrix} = -\sigma^x S^{+}$$

$$S^{-}S^{+} + S^{+}S^{-} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$S^{-}S^{-} = S^{+}S^{+} = 0$$

This means that, now we have two groups of field operator $\{\sigma^x\}, \{S^{-}, S^{+}\}$. For group $\{\sigma^x\}$, they are totally bosonic. For group $\{S^{-}, S^{+}\}$, they are locally fermionic and bosonic for different sites. By noticing operators between those groups have local fermionic relationship, we can rebuild a group containing pure fermionic operators.

$$c_i^{\dagger} = \prod_{l<i} \sigma_l^x \, S_i^{+}$$

$$c_j = \prod_{k<j} \sigma_k^x \, S_j^{-}$$

It is easy to verify $\{c^{\dagger}, c\}$ is a group of fermionic operators by verify anticommutation relationship.

Of course, we can also describe $\{S^{-}, S^{+}\}$ by $\{c^{\dagger}, c\}$ and $\{\sigma^x\}$,

$$S_i^{+} = \prod_{l<i} \sigma_l^x \, c_i^{\dagger}$$

$$S_j^{-} = \prod_{k<j} \sigma_k^x \, c_j$$

Now we can rewrite our bosonic Hamiltonians by pure fermionic operators.

$$H_1 = \sum_{j=1}^{n} [2K_1^{*}(c_j^{\dagger}c_j - \frac{1}{2})]$$

$$H_2 = \sum_{j=1}^{n} \left[K_2\left(c_j^{\dagger}c_{j+1} + c_{j+1}^{\dagger}c_j + c_j^{\dagger}c_{j+1}^{\dagger} + c_{j+1}c_j\right)\right]$$

For Hamiltonian 1, it is already diagonalized. But for 2, it is not the case.

So, we need do Fourier transformation to all of them. The transformation relationship can be written below,

$$c_j = \frac{1}{\sqrt{n}} \sum_k e^{ikx_j} c_k$$

$$c_j^\dagger = \frac{1}{\sqrt{n}} \sum_k e^{-ikx_j} c_k^\dagger$$

After Fourier transformation, those Hamiltonians are,

$$H_1 = \sum_{|k|} 2K_1^* (c_k^\dagger c_k + c_{-k}^\dagger c_{-k} - 1)$$

$$H_2 = \sum_{|k|} \left[2K_2 \left(\cos(k)(c_k^\dagger c_k + c_{-k}^\dagger c_{-k}) + i sin(k)(c_k^\dagger c_{-k}^\dagger - c_{-k} c_k) \right) \right]$$

We notice that for those Hamiltonians, if we combine k and $-k$ terms together, they will all be partially diagonalized according to $|k|$. We would write partially diagonalized parts of those Hamiltonians in 2×2 matrixes with single particle basis below,

$$\begin{bmatrix} c_k^\dagger & c_{-k} \end{bmatrix} H(|k|)_i \begin{bmatrix} c_k \\ c_{-k}^\dagger \end{bmatrix}$$

Now, $H(|k|)_i$ in matrix form are,

$$H(|k|)_1 = 2K_1^* \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$H(|k|)_2 = 2K_2 (\begin{bmatrix} \cos(k) & i \cdot \sin(k) \\ -i \cdot \sin(k) & -\cos(k) \end{bmatrix} + \cos(k))$$

And $V(|k|)_1 = e^{H(|k|)_1}, V(|k|)_2 = e^{H(|k|)_2}$,

$$V(|k|)_1 = \begin{bmatrix} e^{2K_1^*} & 0 \\ 0 & e^{-2K_1^*} \end{bmatrix}$$

$$V(|k|)_2 = e^{2K_2 \cos(k)} \begin{bmatrix} \cosh(2K_2) + \sinh(2K_2)\cos(k) & i \cdot \sinh(2K_2)\sin(k) \\ -i \cdot \sinh(2K_2)\sin(k) & \cosh(2K_2) - \sinh(2K_2)\cos(k) \end{bmatrix}$$

Then we will diagonalize $V(|k|)_1 V(|k|)_2$.

$$(A - E)(C - E) = B$$

$$E = \frac{(A + C) \pm \sqrt{(A + C)^2 - 4(AC - B)}}{2}$$

Here,

$$A = e^{2K_1^*}(\cosh(2K_2) + \sinh(2K_2)\cos(k))$$

$$B = (\sinh(2K_2)\sin(k))^2$$

$$C = e^{-2K_1^*}(\cosh(2K_2) - \sinh(2K_2)\cos(k))$$

We notice that,

$$AC - B = 1$$

$$A + C = 2\cosh(2K_1^*)\cosh(2K_2) + 2\sinh(2K_1^*)\sinh(2K_2)\cos(k) \geq 2$$

So, there are two positive eigenvalues, we assume they are $E_1 = e^{\varepsilon_1}, E_2 = e^{\varepsilon_2}$. The reason we do that is now we diagonalized this matrix in single particle basis. And now we want to get the eigenvalues in many-particle basis. Remember for a certain one-particle Hamiltonian operator, many-particle eigenvalues are just different pluses between single particle eigenvalues. Now the one-particle Hamiltonian operator is in exponential part, so eigenvalues in many-particle basis should be products between different single particle eigenvalues.

By noticing $E_1 \cdot E_2 = 1$, we have 4 eigenvalues of $V(|k|)_1 V(|k|)_2$ in many-particle basis,

$$E_{1-} = e^{2K_2\cos(k)}e^{-\varepsilon_k}$$
$$E_0 = e^{2K_2\cos(k)}$$
$$E_2 = e^{2K_2\cos(k)}$$
$$E_{1+} = e^{2K_2\cos(k)}e^{\varepsilon_k}$$

Here, $cosh(\varepsilon_k) = \frac{E_1+E_2}{2} = \frac{A+C}{2} = \cosh(2K_1^*)\cosh(2K_2) + \sinh(2K_1^*)\sinh(2K_2)\cos(k)$.

E_{1-}, E_0, E_2, E_{1+} represent eigenvalues of 1, 0, 2, 1 particle respectively.

Now, finally we can express the partition function we write from very beginning explicitly.

$$\begin{aligned} Z &= Tr((V_1V_2)^m) \\ &= \lim_{m\to\infty} \lambda_{max}^m \\ &= \lim_{m\to\infty} \left((2\sinh(2K_1))^{\frac{n}{2}} e^{\sum_{|k|}(2K_2\cos(k)+\varepsilon_k)}\right)^m \\ &= \lim_{n\to\infty}\lim_{m\to\infty} \left((2\sinh(2K_1))^{\frac{n}{2}} e^{\frac{n}{2\pi}\int_0^\pi(2K_2\cos(k)+\varepsilon_k)dk}\right)^m \end{aligned}$$

If we write down free energy per-site, we have

$$\begin{aligned} F &= -kT\frac{\ln(Z)}{mn} \\ &= -kT[\frac{1}{2}\ln(2\sinh(2K_1)) + \frac{1}{2\pi}\int_0^\pi \varepsilon_k dk] \end{aligned}$$

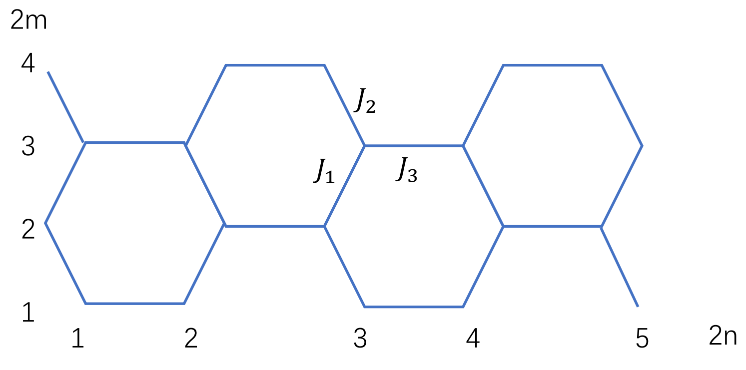
By using identical condition,

$$\varepsilon_k = \frac{1}{\pi}\int_0^\pi \ln(2\cosh(\varepsilon_k) + 2\cos(w))dw$$

The expression of F by using K_1, K_2 is,

$$F = -kT\{\ln(2) + \frac{1}{2\pi^2}\int_0^\pi\int_0^\pi \ln[\cosh(2K_1)\cosh(2K_2) + \sinh(2K_2)\cos(k) + \sinh(2K_1)\cos(w)]\,dkdw\}$$

2D Honeycomb Lattice



P. 1

Now we would like to solve an 2D Ising model in honeycomb lattice shown in P.1, whose Hamiltonian is, (Assuming $J_1, J_2, J_3 > 0$)

$$H = -\sum_{i=1}^{m}\sum_{j=1}^{n}(J_1\sigma_{2i,2j-1}^z\sigma_{2i+1,2j-1}^z + J_1\sigma_{2i-1,2j}^z\sigma_{2i,2j}^z$$

$$+J_2\sigma_{2i-1,2j-1}^z\sigma_{2i,2j-1}^z + J_2\sigma_{2i,2j}^z\sigma_{2i+1,2j}^z$$

$$+J_3\sigma_{2i-1,2j-1}^z\sigma_{2i-1,2j}^z + J_3\sigma_{2i,2j}^z\sigma_{2i,2j+1}^z)$$

The main method is similar with the one above. Partition function is,

$$Z = \sum_{\{\sigma^z\}} e^{-\beta H}$$

By rewriting $K_i = \beta J_i$, we have

$$Z = \sum_{\{\sigma^z\}} \left\langle 1 \middle| e^{\sum_{j=1}^{n}(K_1\sigma_{1,2j}^z\sigma_{2,2j}^z + K_2\sigma_{1,2j-1}^z\sigma_{2,2j-1}^z + K_3\sigma_{1,2j-1}^z\sigma_{1,2j}^z)} \middle| 2 \right\rangle$$

$$\cdot \left\langle 2 \middle| e^{\sum_{j=1}^{n}(K_1\sigma_{2,2j-1}^z\sigma_{3,2j-1}^z + K_2\sigma_{2,2j}^z\sigma_{3,2j}^z + K_3\sigma_{2,2j}^z\sigma_{2,2j+1}^z)} \middle| 3 \right\rangle$$

$$\cdots$$

$$\cdot \langle 2m | \cdots | 1 \rangle$$

Here $\langle 1|A_{12}|2\rangle$ means elements of matrix A_{12} can be determined by $2^{2n} \times 2^{2n}$ combinations of variables $\sigma_{1,j}^z$ and $\sigma_{2,j}^z$.

If we define $2^{2n} \times 2^{2n}$ matrixes V_1, V_2, V_3 and V_4 as below,

$$V_1 = e^{\sum_{j=1}^{n}(K_1\sigma_{2j}^z{\sigma'}_{2j}^z + K_2\sigma_{2j-1}^z{\sigma'}_{2j-1}^z)}$$

$$V_2 = e^{\sum_{j=1}^{n}K_3\sigma_{2j-1}^z\sigma_{2j}^z}\prod_{i=1}^{2n}\delta_{\sigma_i^z,{\sigma'}_i^z}$$

$$V_3 = e^{\sum_{j=1}^{n}(K_1\sigma_{2j-1}^z{\sigma'}_{2j-1}^z + K_2\sigma_{2j}^z{\sigma'}_{2j}^z)}$$

$$V_4 = e^{\sum_{j=1}^{n}K_3\sigma_{2j}^z\sigma_{2j+1}^z}\prod_{i=1}^{2n}\delta_{\sigma_i^z,{\sigma'}_i^z}$$

We can rewrite partition function in a simple way,

$$Z = Tr[(V_1V_2V_3V_4)^m]$$

What we need to do for next step is diagonalize matrix $V_1V_2V_3V_4$, and $Z = \sum_{i=1}^{2^{2n}}\lambda_i^m$.(Here λ_i is the eigenvalue of $V_1V_2V_3V_4$ labeled by i)

We see the exponential part of V_i matrixes as Hamiltonians, by using the trick we introduced above again, those 4 Hamiltonian are,

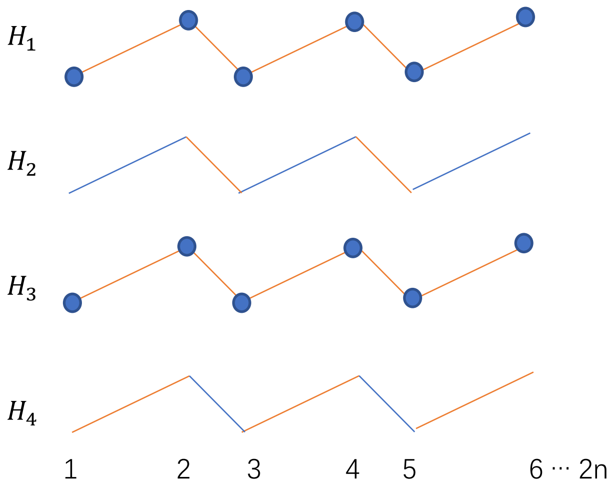
$$H_1=\sum_{j=1}^{n}(K_1^*\sigma_{2j}^x+K_2^*\sigma_{2j-1}^x)$$

$$H_2=\sum_{j=1}^{n}K_3\sigma_{2j-1}^z\sigma_{2j}^z$$

$$H_3=\sum_{j=1}^{n}(K_1^*\sigma_{2j-1}^x+K_2^*\sigma_{2j}^x)$$

$$H_4=\sum_{j=1}^{n}K_3\sigma_{2j}^z\sigma_{2j+1}^z$$

And those Hamiltonians just describe models shown in P.2 below.



P. 2

Here blue points and lines show interactions. Now we rewrite our bosonic Hamiltonians by pure fermionic operators.

$$H_1=\sum_{j=1}^{n}[2K_1^*c_{2j}^\dagger c_{2j}+2K_2^*c_{2j-1}^\dagger c_{2j-1}-K_1^*-K_2^*]$$

$$H_2=\sum_{j=1}^{n}\left[K_3\left(c_{2j-1}^\dagger c_{2j}+c_{2j}^\dagger c_{2j-1}+c_{2j-1}^\dagger c_{2j}^\dagger+c_{2j}c_{2j-1}\right)\right]$$

$$H_3=\sum_{j=1}^{n}[2K_2^*c_{2j}^\dagger c_{2j}+2K_1^*c_{2j-1}^\dagger c_{2j-1}-K_1^*-K_2^*]$$

$$H_4=\sum_{j=1}^{n}\left[K_3\left(c_{2j}^\dagger c_{2j+1}+c_{2j+1}^\dagger c_{2j}+c_{2j}^\dagger c_{2j+1}^\dagger+c_{2j+1}c_{2j}\right)\right]$$

For Hamiltonian 1 and 3, they are already diagonalized. But for 2 and 4, they are partly diagonalized and not diagonalized at the same basis. (For example, H_2 is partly diagonalized at 1,2 sites but H_4 is partly diagonalized at 2,3 sites. They have an overlap index 2 which makes them do not commute with each other)

So, we need do Fourier transformation to all of them. The transformation relationship can be written below,

$$c_{2j-1} = \frac{1}{\sqrt{n}} \sum_{k} e^{ikx_j} c_{k,A}$$

$$c_{2j} = \frac{1}{\sqrt{n}} \sum_{k} e^{ikx_j} c_{k,B}$$

$$c_{2j-1}^{\dagger} = \frac{1}{\sqrt{n}} \sum_{k} e^{-ikx_j} c_{k,A}^{\dagger}$$

$$c_{2j}^{\dagger} = \frac{1}{\sqrt{n}} \sum_{k} e^{-ikx_j} c_{k,B}^{\dagger}$$

After Fourier transformation, those Hamiltonians are,

$$H_1 = \sum_{k} \left[2K_1^* (c_{k,B}^{\dagger} c_{k,B} - \frac{1}{2}) + 2K_2^* (c_{k,A}^{\dagger} c_{k,A} - \frac{1}{2}) \right]$$

$$H_2 = \sum_{k} \left[K_3 \left(c_{k,A}^{\dagger} c_{k,B} + c_{k,B}^{\dagger} c_{k,A} + c_{k,A}^{\dagger} c_{-k,B}^{\dagger} + c_{-k,B} c_{k,A} \right) \right]$$

$$H_3 = \sum_{k} \left[2K_2^* (c_{k,B}^{\dagger} c_{k,B} - \frac{1}{2}) + 2K_1^* (c_{k,A}^{\dagger} c_{k,A} - \frac{1}{2}) \right]$$

$$H_4 = \sum_{k} \left[K_3 \left(e^{ik} c_{k,B}^{\dagger} c_{k,A} + e^{-ik} c_{k,A}^{\dagger} c_{k,B} + e^{-ik} c_{-k,B}^{\dagger} c_{k,A}^{\dagger} + e^{ik} c_{k,A} c_{k,B} \right) \right]$$

Again, write partially diagonalized parts of those Hamiltonians in 4×4 matrixes with single particle basis below,

$$\begin{bmatrix} c_{k,A}^{\dagger} & c_{k,B}^{\dagger} & c_{-k,A} & c_{-k,B} \end{bmatrix} H(|k|)_i \begin{bmatrix} c_{k,A} \\ c_{k,B} \\ c_{-k,A}^{\dagger} \\ c_{-k,B}^{\dagger} \end{bmatrix}$$

Now, $H(|k|)_i$ in matrix form are,

$$H(|k|)_1 = \begin{bmatrix} 2K_2^* & 0 & 0 & 0 \\ 0 & 2K_1^* & 0 & 0 \\ 0 & 0 & -2K_2^* & 0 \\ 0 & 0 & 0 & -2K_1^* \end{bmatrix}$$

$$H(|k|)_2 = K_3 \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

$$H(|k|)_3 = \begin{bmatrix} 2K_1^* & 0 & 0 & 0 \\ 0 & 2K_2^* & 0 & 0 \\ 0 & 0 & -2K_1^* & 0 \\ 0 & 0 & 0 & -2K_2^* \end{bmatrix}$$

$$H(|k|)_4 = K_3 \begin{bmatrix} 0 & e^{-ik} & 0 & -e^{-ik} \\ e^{ik} & 0 & e^{ik} & 0 \\ 0 & e^{-ik} & 0 & -e^{-ik} \\ -e^{ik} & 0 & -e^{ik} & 0 \end{bmatrix}$$

We would like to derive the modulus of eigenvalues which is larger than 1 (we call e^{η_k}) for matrix $e^{H(|k|)_1}e^{H(|k|)_2}e^{H(|k|)_3}e^{H(|k|)_4}$. In this way, the product of those eigenvalues $e^{\sum_{|k|}\eta_k}$ can give the largest modulus λ_{max}. Remember λ_{max} should be real and positive, then $e^{\sum_{|k|}\eta_k} = e^{Re[\sum_{|k|}\eta_k]}$.

Actually, it is not straightforward to solve eigenvalues of $e^{H(|k|)_1}e^{H(|k|)_2}e^{H(|k|)_3}e^{H(|k|)_4}$. Fortunately, we do not have to solve all of them in our final result, by noticing the eigenvalues of $e^{H(|k|)_1}e^{H(|k|)_2}e^{H(|k|)_3}e^{H(|k|)_4}$ have a character below,

$$E_1 \times E_2^* = 1$$
$$E_3 \times E_4^* = 1$$
$$E_1 \times E_2 \times E_3 \times E_4 = 1$$

So, we can rewrite 4 eigenvalues E_1, E_2, E_3, E_4 as $e^{\eta_{k,1}}, e^{-\eta_{k,1}^*}, e^{\eta_{k,2}}, e^{-\eta_{k,2}^*}$. Then partition function is,

$$\begin{aligned} Z &= Tr[(V_1V_2V_3V_4)^m] \\ &= \lim_{m\to\infty} \lambda_{max}^m \\ &= \lim_{m\to\infty} \left((2\sinh(2K_1))^n(2\sinh(2K_2))^n e^{Re[\sum_{|k|}(\eta_{k,1}+\eta_{k,2})]}\right)^m \\ &= \lim_{n\to\infty}\lim_{m\to\infty} \left((4\sinh(2K_1)\sinh(2K_2))^n e^{Re\left[\frac{n}{4\pi}\int_0^{2\pi}(\eta_{k,1}+\eta_{k,2})dk\right]}\right)^m \end{aligned}$$

Free energy per-site is,

$$\begin{aligned} F &= -kT\frac{\ln(Z)}{4mn} \\ &= -kT\left(\frac{1}{4}\ln(4\sinh(2K_1)\sinh(2K_2)) + \frac{1}{32\pi}\int_0^{2\pi}\left(\eta_{k,1}+\eta_{k,2}+\eta_{k,1}^*+\eta_{k,2}^*\right)dk\right) \\ &= -kT\{\frac{1}{4}\ln(\sinh(2K_1)\sinh(2K_2)) + \frac{1}{2}\ln(2) \\ &\quad + \frac{1}{64\pi^2}\int_0^{2\pi}dk\int_0^{2\pi}\ln\left(\prod_i^4(e^{\eta_{k,i}}+e^{-\eta_{k,i}}-2\cos(w))\right)dw\} \end{aligned}$$

Now we write down eigenequation of $e^{H(|k|)_1}e^{H(|k|)_2}e^{H(|k|)_3}e^{H(|k|)_4}$ explicitly,

$$E^4 + (A_r + iA_i)E^3 + B_rE^2 + (A_r - iA_i)E + 1 = 0$$

Here,

$$A_r = -4\cosh(2K_1^*)\cosh(2K_2^*)\cosh(2K_3) - 2\sinh(2K_1^*)\sinh(2K_2^*)\left(1+\cosh^2 2K_3\right)$$

$$
\begin{aligned}
&-(\sinh^2(2K_1^*) + \sinh^2(2K_2^*)) \sinh^2(2K_3) \cos(k) \\
A_i &= (\sinh^2(2K_1^*) - \sinh^2(2K_2^*)) \sinh^2(2K_3) \sin(k) \\
B_r &= 2\cosh^2(2K_3) \\
&+2(\cosh^2(2K_1^*) \cosh^2(2K_2^*) + \sinh^2(2K_1^*) \sinh^2(2K_2^*))(1 + \cosh^2(2K_3)) \\
&+8\sinh(2K_1^*) \sinh(2K_2^*) \cosh(2K_1^*) \cosh(2K_2^*) \cosh(2K_3) \\
&-4\sinh(2K_1^*) \sinh(2K_2^*) \sinh^2(2K_3) \cos(k)
\end{aligned}
$$

We make it a reciprocal 8th degree equation, whose roots are $E_1, \frac{1}{E_1^*}, E_2, \frac{1}{E_2^*}, \frac{1}{E_1}, E_1^*, \frac{1}{E_2}, E_2^*$

$$
E^8 + 2\mathrm{A_r}E^7 + (2B_r + A_r^2 + A_i^2)E^6 + 2\mathrm{A_r}(1 + \mathrm{B_r})\mathrm{E}^5 + (2 + 2\mathrm{A_r^2} - 2\mathrm{A_i^2} + \mathrm{B_r^2})\mathrm{E}^4 + 2\mathrm{A_r}(1 + \mathrm{B_r})\mathrm{E}^3 + (2B_r + A_r^2 + A_i^2)\mathrm{E}^2 + 2\mathrm{A_r}\mathrm{E} + 1 = 0
$$

Consider a general reciprocal 8th degree equation,

$$
x^8 + ax^7 + bx^6 + cx^5 + dx^4 + cx^3 + bx^2 + ax + 1 = 0
$$

By replacing $x + \frac{1}{x} = y$, we have,

$$
y^4 + ay^3 + (b-4)y^2 + (c-3a)y + d - 2b + 2 = 0
$$

We would like to get a result $(y_1 - m)(y_2 - m)(y_3 - m)(y_4 - m)$. (Here, y_1, y_2, y_3, y_4 are 4 roots of this equation). By using Vieta theorem for 4th degree equation,

$$
\begin{aligned}
&y_1 + y_2 + y_3 + y_4 = -a \\
&y_1y_2 + y_1y_3 + y_1y_4 + y_2y_3 + y_2y_4 + y_3y_4 = b - 4 \\
&y_1y_2y_3 + y_1y_2y_4 + y_1y_3y_4 + y_2y_3y_4 = 3a - c \\
&y_1y_2y_3y_4 = d - 2b + 2
\end{aligned}
$$

Then $X = (y_1 - m)(y_2 - m)(y_3 - m)(y_4 - m)$ can be expressed,

$$
X = d - 2b + 2 - m(3a - c) + m^2(b-4) + m^3a + m^4
$$

Replace a, b, c, d, m by $A_r, A_i, B_r, \cos(w)$

$$
\begin{aligned}
X &= B_r^2 - 4B_r - 4A_i^2 + 4 \\
&-2\cos(w)(4A_r - 2A_rB_r) \\
&+4\cos^2(w)(A_r^2 + A_i^2 + 2B_r - 4) \\
&+8\cos^3(w)\, 2A_r \\
&+16\cos^4(w)
\end{aligned}
$$

Free energy per-site as a function of $X(w,k)$ is,

$$
F = -kT\{\frac{1}{2}\ln(2) + \frac{1}{64\pi^2}\int_0^{2\pi} dk \int_0^{2\pi} \ln[(\sinh(2K_1)\sinh(2K_2))^4 \cdot X(w,k)]\, dw\}
$$

Factorize $X(w,k)$ to X_1, X_2. By knowing,

$$
\begin{aligned}
X_1 - X_2 &= 4(\sinh^2(2K_1^*) - \sinh^2(2K_2^*)) \sinh^2(2K_3) \sin(k) \sin(w) \\
X_1 + X_2 &= 8(\cosh(2K_1^*) \cosh(2K_2^*) \cosh(2K_3) + \sinh(2K_1^*) \sinh(2K_2^*) - \cos(w))^2 \\
&-8(\cos(w) + \cos(k))(\sinh(2K_1^*) \sinh(2K_2^*) \sinh(2K_3)) \\
&-4\cos(k)\cos(w)(\sinh^2(2K_1^*) + \sinh^2(2K_2^*)) \sinh^2(2K_3) \\
&-4(\sinh^2(2K_1^*) + \sinh^2(2K_2^*)) \sinh^2(2K_3)
\end{aligned}
$$

And according to symmetry of w, $\int_0^{2\pi} \ln(X_1) dw = \int_0^{2\pi} \ln(X_2) dw$, so we can rewrite free energy per-site,

$$F = -kT\{\frac{1}{2}\ln(2)$$
$$+\frac{1}{64\pi^2}\int_0^{2\pi} dk \int_0^{2\pi} \ln[(\sinh(2K_1)\sinh(2K_2))^4 \cdot X_1^2]\, dw\}$$
$$= -kT\{\frac{1}{2}\ln(2)$$
$$+\frac{1}{32\pi^2}\int_0^{2\pi} dk \int_0^{2\pi} \ln[(\sinh(2K_1)\sinh(2K_2))^2 \cdot X_1]\, dw\}$$

Explicitly, factorize $(\sinh(2K_1)\sinh(2K_2))^2 \cdot X_1$ again,

$$4\Big(\cosh(2K_1)\cosh(2K_2)\cosh(2K_3) + 1 - \cos(w)\sinh(2K_1)\sinh(2K_2)$$
$$+ \sinh(2K_1)\sinh(2K_3)\cos\left(\frac{w-k}{2}\right) + \sinh(2K_1)\sinh(2K_3)\cos\left(\frac{w+k}{2}\right)\Big)$$
$$\cdot (\cosh(2K_1)\cosh(2K_2)\cosh(2K_3) + 1 - \cos(w)\sinh(2K_1)\sinh(2K_2)$$
$$- \sinh(2K_1)\sinh(2K_3)\cos\left(\frac{w-k}{2}\right) - \sinh(2K_1)\sinh(2K_3)\cos(\frac{w+k}{2}))$$

By replacing $\frac{w-k}{2}, \frac{w+k}{2}$ with w_1, w_2, notice those two terms are equal again according to symmetry of w_1, w_2

$$F = -kT\{\frac{3}{4}\ln(2)$$
$$+\frac{1}{16\pi^2}\int_0^{2\pi}\int_0^{2\pi} \ln[(\cosh(2K_1)\cosh(2K_2)\cosh(2K_3) + 1$$
$$- \cos(w_1 + w_2)\sinh(2K_1)\sinh(2K_2) - \sinh(2K_1)\sinh(2K_3)\cos(w_1)$$
$$- \sinh(2K_1)\sinh(2K_3)\cos(w_2))]\, dw_1 dw_2\}$$

It has a very similar form with the case in 2D square lattice.

Discussion

Finally, as an end of this paper, we make the term in ln ( ) of free energy per-site zero, give phase transition conditions for those two models.

$$\cosh(2K_1)\cosh(2K_2) - \sinh(2K_1) - \sinh(2K_2) = 0$$
$$\rightarrow \sinh\left(\frac{2J_1}{kT_c}\right)\sinh\left(\frac{2J_2}{kT_c}\right) = 1$$

And

$$\cosh(2K_1)\cosh(2K_2)\cosh(2K_3) + 1 - \sinh(2K_1)\sinh(2K_2) - \sinh(2K_1)\sinh(2K_3)$$
$$- \sinh(2K_1)\sinh(2K_3) = 0$$
$$\rightarrow \sinh\left(\frac{2J_1}{kT_c}\right)\sinh\left(\frac{2J_2}{kT_c}\right)\sinh\left(\frac{2J_3}{kT_c}\right) = \sinh\left(\frac{2J_1}{kT_c}\right) + \sinh\left(\frac{2J_2}{kT_c}\right) + \sinh(\frac{2J_3}{kT_c})$$

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